Fast deconvolution with approximated PSF by RSTLS with antireflective boundary conditions

Marco Donatelli a,∗, Nicola Mastronardi b

a Dipartimentodi Scienza e Alta Tecnologia, Università dell’Insubria, Via Valleggio 11, 22100, Como, Italy
b Istituto per le Applicazioni del Calcolo “M. Picone”, sede di Bari, Consiglio Nazionale delle Ricerche, Via G. Amendola, 122/D, I-70126 Bari, Italy

ARTICLE INFO

Article history:
Received 7 December 2011
Received in revised form 19 March 2012

Keywords:
Deconvolution
Structured total least squares
Antireflective boundary conditions

ABSTRACT

The problem of reconstructing signals and images from degraded ones is considered in this paper. The latter problem is formulated as a linear system whose coefficient matrix models the unknown point spread function and the right hand side represents the observed image. Moreover, the coefficient matrix is very ill-conditioned, requiring an additional regularization term. Different boundary conditions can be proposed. In this paper antireflective boundary conditions are considered. Since both sides of the linear system have uncertainties and the coefficient matrix is highly structured, the Regularized Structured Total Least Squares approach seems to be the more appropriate one to compute an approximation of the true signal/image. With the latter approach the original problem is formulated as an highly nonconvex one, and seldom can the global minimum be computed. It is shown that Regularized Structured Total Least Squares problems for antireflective boundary conditions can be decomposed into single variable subproblems by a discrete sine transform. Such subproblems are then transformed into one-dimensional unimodal real-valued minimization problems which can be solved globally. Some numerical examples show the effectiveness of the proposed approach.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

The problem of reconstructing images from degraded ones is considered in this paper. The discrete mathematical model of a one- or multi-dimensional blurred image gives rise to a matrix equation

Af = g,

(1)

where the blurring $A$ of the unknown true image $f$ is modeled with the discrete point spread function (PSF), that we assume to be spatially invariant, and the imposed boundary conditions (BCs) and $g$ is the observed image [1]. In this paper we restrict ourself to a quadrantly symmetric (i.e., symmetric in both horizontal and vertical directions) blur like a Gaussian blur that often appears in applications.

Usually the observed image $g$ is affected by noise. Moreover, if the cause of the blur is not known exactly, then the matrix $A$ in (1) is also uncertain, and the problem is known as blind deconvolution. In this case (1) should be replaced by the following problem,

$(A + E)f = g + \epsilon$,

(2)
with $E$ and $\epsilon$ unknown. Our goal is to compute an approximation of $f$ given an approximation $A$ of the true blurring matrix and the observed image $g$ that is blurred and noisy.

The blurring process produces a blurred image $g$ that at the boundary depends also on entries of the true image $f$ that are outside of the field of view. A common approach to deal with such underdetermination is the application of appropriate BCs. Classical choices are zero Dirichlet, periodic and reflective BCs [1]. Independently on the BCs the matrix $A$ is severely ill-conditioned with singular values decaying to zero without a significant gap to indicate the numerical rank. The spatially invariant assumption for one-dimensional problems implies that the structure of $A$ is Toeplitz (the entries are constant along the diagonals), up to the low rank perturbation due the BCs. For multi-dimensional problems the same structure appears at each level, for instance for bi-dimensional problems (the diagonals), up to the low rank perturbation due the BCs. Formulti-dimensional problemsthesamestructureappearsat invariant assumptionforone-dimensionalproblemsimpliesthatthestructureof ill-conditionedwithsingularvaluesdecayingtozerowithoutsSIGNificantgaptoindicatethenumerical rank. The spatially

A system $(A+E)f = g + \epsilon$ is compatible, i.e.,

$$\min_{f,\epsilon} \|E\|^2 + \|\epsilon\|^2$$

s.t. $(A+E)f = g + \epsilon$.

Although the latter problem is nonconvex, in the so called generic case, it can be solved efficiently and globally by computing the right singular vector associated to the smallest singular value of the augmented matrix $[A, g]$ (see, i.e., [2]). If the matrix $A$ has a structure (Toeplitz, Hankel, block-Toeplitz, ...), it is natural to require that the correction matrix $E$ inherits the same structure. With the structured total least squares problem (STLS) [3–5], sometimes called constrained total least squares [6], the correction matrix $E$ and the vector $\epsilon$ with minimal sum of squared norms are sought, such that the system $(A+E)f = g + \epsilon$ is compatible and $E$ has the same structure of $A$, i.e.,

$$\min_{f,\epsilon} \|E\|^2 + \|\epsilon\|^2$$

s.t. $(A+E)f = g + \epsilon$

$A$ and $E$ same structure.

The STLS problem is a nonconvex problem, and thus finding its global solution is in general a difficult task and some iterative methods computing a local minimum have been proposed [4,5,7,8]. The global solution can be computed in a few cases. For block circulant structures with unstructured blocks the corresponding STLS problem can be solved by decomposing the problem into several smaller TLS problems using the discrete Fourier transform [9].

Regularization has been added to the STLS problem to compute an approximate solution in case the matrix $A$ is ill-conditioned [10,7,11,12] resulting in the following RSTLS problem involving a regularization matrix $L$.

$$\min_{f,\epsilon} \|E\|^2 + \|\epsilon\|^2 + \rho \|f\|^2$$

s.t. $(A+E)f = g + \epsilon$

$A$ and $E$ same structure.

Common choices for $L$ are the identity matrix or a matrix approximating the first or the second derivative operator [1]. For periodic or reflective BCs, the latter requiring a quadrantal symmetric blur, it has been shown in [11] that, since the involved matrices in the RSTLS problem are simultaneously diagonalizable by a unitary transform, fast Fourier transform (FFT) or discrete cosine transform (DCT), the equations can be decoupled allowing to compute the global minimum of univariate real functions.

Antireflective BCs, introduced in [13], have been proven to be more appropriate for a moderate level of noise, giving better reconstructions compared to those obtained imposing reflective BCs [14–16]. See [17] for a review on Antireflective BCs. Unfortunately, the matrix obtained imposing the antireflective BCs is not diagonalizable by unitary transformations also in the case of quadrantal symmetric blur. Nevertheless, it is possible to transform it into a simple matrix by means of trigonometric transforms by using two different approaches described in [16,18]. These approaches are based on the fact that the eigenvector basis of an antireflective matrix of order $n$ is made by $n-2$ sin frequency functions and 2 more linear functions. In particular, the latter linear functions are low frequencies associated to the eigenvalue one of multiplicity two. Exploiting this property, in [16] the components corresponding to the linear eigenvectors are first reconstructed. Then the remaining problem of order $n-2$, after a manipulation, is hence diagonalized by the discrete sine transform. A nonunitary antireflective transform, allowing the extension of the implementation of several filtering-based regularization methods has been proposed in [18]. Antireflective BCs have been successfully applied to iterative regularization methods [14], truncated singular values decomposition [15], Tikhonov regularization [16,18,19] and to fixed point iterations for Total Variation [20].

Other techniques that reduce the effects of ringing generated by an inexact reconstruction in the boundary of an image have been proposed in [21–23]. With such approaches, fast transforms can be used only for computing the matrix-vector
products. Hence these techniques are suitable when the approximate solution is computed with iterative regularizing methods and can not be used as a Tikhonov-style deconvolution technique in a direct way.

The RSTLS method [11] for reconstructing images, given an approximation of the PSF, (2) with antireflective BCs will be considered in this paper. Two methods will be proposed following the strategies introduced in [16,18] for Tikhonov regularization. It will be shown that the two proposed methods compute the global solution of the involved nonconvex optimization problem and give comparable numerical results. In particular, the approach based on blurring (introduced in [24]) will be analyzed in detail also for multi-dimensional problems. Such an approach, applied to a Tikhonov filtering-based regularization method, has been extensively investigated in [18]. The antireflective transform (ART), introduced in [18], related to the discrete sine transform, allows one to work in the frequency domain, similarly to the FFT and the DCT. The drawback is that the ART transform is not a unitary one. In order to use ART with the RSTLS approach, the $\ell_2$-norm must be replaced by a “new” norm such that the matrix associated to ART works as a diagonal matrix decoupling the equations. In such a way, an algorithm can be derived simply replacing the FFT or the DCT with the ART and this can be quite easily extended to multidimensional problems. Since the computational costs of ART, DCT and FFT are the same, the proposed algorithm has the same computational complexity of the one proposed in [11] for reflective and periodic BCs. Moreover, it allows us to gain more accurate reconstructions of the images due to a better choice of the BCs.

The paper is organized as follows. The antireflective BCs are described in Section 2 together with the properties of the associated matrices and of the ART. The main features of the RSTLS method for matrices diagonalizable by unitary transforms are described in Section 3. Two RSTLS methods for antireflective BCs, inspired from the two different techniques introduced in [16,18] for Tikhonov regularization, are described in Sections 4 and 5, respectively. Some numerical examples for the deconvolution of signals are proposed in Section 6. In Section 7, the approach proposed in Section 5 is extended to bidimensional problems, followed by some numerical experiments and followed by the conclusions.

2. Antireflective boundary conditions

In this section the antireflective BCs are considered, and the spectral decomposition of the coefficient matrix obtained imposing such BCs on the problem is described using the ART. Finally, it is shown how antireflective BCs are used to reconstruct signals with the Tikhonov method.

For brevity we report a one-dimensional description of the BCs model that can be extended to multi-dimensional problems applying the same strategy in each direction and obtaining an eigenvector matrix defined by tensor product (Ref. Section 7). Let $f = (\ldots, f_0, f_1, \ldots, f_n, f_{n+1}, \ldots)^T$ be the true signal and $(j)_{j=1}^m$ the set of indexes in the window of observation of the signal. Given a PSF $h = (h_{-m}, \ldots, h_0, \ldots, h_m)$, with $2m + 1 \leq n$, we can associate the symbol

$$h(x) = \sum_{j=-m}^{m} h_j e^{i j x}, \quad i = \sqrt{-1}. \quad (4)$$

Before introducing the antireflective BCs and the ART matrix, we briefly recall the periodic and reflective BCs.

**Periodic BCs** are defined by imposing periodicity on the signal as follows

$$f_{1-j} = f_{n+1-j} \quad \text{and} \quad f_{n+j} = f_j,$$

for $j = 1, \ldots, n$. The blurring matrix $A_P$ associated to periodic BCs is diagonalized by the Fourier matrix

$$F^{(n)}_{ij} = \frac{1}{\sqrt{n}} \exp \left( -\frac{i 2\pi (i-1)(j-1)}{n} \right), \quad i, j = 1, \ldots, n.$$

More precisely, the blurring matrix is

$$A_P = (F^{(n)})^T \text{diag}(h(x)) F^{(n)}, \quad (5)$$

where $x_i = 2(i - 1)\pi/n$, for $i = 1, \ldots, n$. Notice that its eigenvalues $\lambda_i = h(x_i)$ can be easily computed since $\lambda_i = [F^{(n)}(A_P e_i)] i/[F^{(n)}(e_i)]$, where $e_i$ is the first vector of the canonical basis.

**Reflective BCs** are defined by imposing an even symmetry around the boundary of the signal as follows

$$f_{1-j} = f_j \quad \text{and} \quad f_{n+j} = f_{n+1-j}.$$

for $j = 1, \ldots, n$. If the PSF is symmetric, i.e., $h_{-j} = h_j$, then the blurring matrix associated to reflective BCs is diagonalized by the cosine transform (see [25])

$$C^{(n)}_{ij} = \sqrt{\frac{2 - \delta_{i,1}}{n}} \cos \left( \frac{(i-1)(2j-1)\pi}{2n} \right), \quad i, j = 1, \ldots, n,$$

where $\delta_{i,1} = 1$ if $i = 1$ and zero otherwise. More precisely, the blurring matrix is

$$A_R = (C^{(n)})^T \text{diag}(h(x)) C^{(n)}, \quad (6)$$
where $x_i = (i - 1)\pi/n$, for $i = 1, \ldots, n$. Like for periodic BCs, the eigenvalues of $A_h$ can be easily computed, i.e., $\lambda_i = [C^{(n)}(A_h e_1)]_i/[C^{(n)} e_1]$.

2.1. The antireflective matrices

Antireflective BCs impose a central symmetry around the boundary points by

$$f_{i-j} = f_i - (f_{i+1} - f_i) = 2f_i - f_{i+1},$$
$$f_{n+j} = f_n - (f_{n-j} - f_n) = 2f_n - f_{n-j},$$

for $j = 1, \ldots, m$. Note that reflective BCs preserve the continuity of the signal at the boundary, while antireflective BCs preserve the continuity both of the signal and of its first derivative. Following the analysis given in [13], if $h$ is symmetric and $\sum_{j=-m}^{m} h_j = 1$, the precise structure of the antireflective matrix is

$$A_h = \begin{pmatrix} z_1 + h_0 & 0 & \cdots & 0 & 0 \\
2z_2 + h_1 & 0 & & & \\
\vdots & & & & \\
2z_m + h_{m-1} & \hat{A} & z_m + h_{m-1} & & \\
h_m & & & & \\
0 & 2z_2 + h_1 & & & \\
0 & 0 & \cdots & 0 & z_1 + h_0 \end{pmatrix} \quad (7)$$

where $[A_1]_{1,1} = [A_h]_{n,n} = 1$, $z_j = 2\sum_{i=j}^{m} h_i$, and $\hat{A}$ is a matrix of order $n-2$, diagonalized by the discrete sine transform. More precisely, let $Q^{(n)}$ be the discrete sine transform

$$Q^{(n)}_{ij} = \sqrt{\frac{2}{n+1}} \sin \left( \frac{j\pi}{n+1} \right), \quad i, j = 1, \ldots, n, \quad (8)$$

then

$$\hat{A} = (Q^{(n-2)})^H \text{diag}(h(x)) Q^{(n-2)}, \quad (9)$$

where $x_i = i\pi/(n-1)$, for $i = 1, \ldots, n-2$ (see [26]). The eigenvalues $\lambda_i = h(\alpha_i)$ can be easily computed by $\lambda_i = [Q^{(n-2)}(\hat{A} e_1)]_i/[Q^{(n-2)} e_1]$.

2.2. The antireflective transform

The ART introduced in [18] can be defined by the matrix

$$T_n = \begin{bmatrix} \alpha_n^{-1} & Q^{(n-2)} & \alpha_n^{-1} p \\
\alpha_n^{-1} p^T & 0 & \alpha_n^{-1} \end{bmatrix}, \quad (10)$$

where the vector $\hat{p} = [1, p^T, 0]^T$ is the sampling of the function $1 - x$ on the grid $j/(n-1)$ for $j = 0, \ldots, n-1$, i.e., $p_j = 1 - j/(n-1)$ for $j = 1, \ldots, n-2$,

$$\alpha_n = \|\hat{p}\|_2 = \sqrt{\frac{n(2n-1)}{6(n-1)}}, \quad (11)$$

and $J$ is the $n-2$ dimensional flip matrix, i.e.,

$$J_{s,t} = \begin{cases} 1 & \text{if } s+t = n-1, \\
0 & \text{otherwise}. \end{cases}$$

It turns out [18] that the inverse of the ART is the matrix

$$T_n^{-1} = \begin{bmatrix} \alpha_n^{-1} & Q^{(n-2)} & -Q^{(n-2)} p \\
\alpha_n^{-1} p^T & 0 & \alpha_n \end{bmatrix}. \quad (12)$$

The Jordan canonical form of the antireflective matrix in (7) is

$$A_h = A_n(h) = T_n D_n T_n^{-1}, \quad (13)$$

where $D_n = \text{diag}(h(x))$ with $x$ defined as $x_1 = x_n = 0$ and $x_{j+1} = \frac{j\pi}{n-1}$ for $j = 1, \ldots, n-2$. 

\[ \text{Author's personal copy} \]
The factorization (13) can be very useful to implement spectral filtering methods with the reblurring strategy [18]. Moreover, from (13) it is easy to prove that the antireflective matrices define an algebra. Such an algebra is not closed by transposition, an operation that in the case of antireflective BCs could generate artifacts at the edges. From a computational point of view, the antireflective transform and its inverse can be computed with \( O(n \log n) \) real floating point operations [13].

2.3. Tikhonov regularization with antireflective BCs

The Tikhonov regularization method computes the solution of the following minimization problem

\[
\min_{f \in \mathbb{R}^n} \| A f - g \|^2 + \rho \| f \|_1, \quad \rho > 0.
\]  

(14)

We assume that the matrices \( A \) and \( L \) satisfy

\[ \mathcal{N}(A) \cap \mathcal{N}(L) = \{0\}, \]

(15)

where \( \mathcal{N}(M) \) denotes the null space of the matrix \( M \). Under such an assumption the minimization problem (14) has a unique solution that is also the unique solution of the linear system

\[
(A^T A + \rho L^T L) f = A^T g.
\]  

(16)

For antireflective BCs in both \( A \) and \( L \), since \( A_n^T \neq A_n \) even for a symmetric PSF and the antireflective algebra is not closed under transposition, instead of solving (16) the reblurring equation to be solved is

\[
A_n(h^2 + \rho \ell^2) f = A_n(h) g,
\]  

(17)

where \( \ell(x) \) is the symbol of the matrix \( L = A_n(\ell) \), see [24,19]. A class of regularization methods is obtained through spectral filtering [1]. Specifically, if the spectral decomposition of \( A \) is

\[
A = T_n \text{diag}(d) \, T_n^{-1}, \quad T_n = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}, \quad T_n^{-1} = \begin{bmatrix} t_1^{-1} & t_2^{-1} & \cdots & t_n^{-1} \end{bmatrix},
\]

then a spectral filter solution is given by

\[
f_{\text{reg}} = \sum_{i=1}^n \phi_i \hat{t}_i g, \]

(18)

where \( \phi_i \) are filter factors that satisfy

\[
\phi_i \approx \begin{cases} 1 & \text{if } d_i \text{ is large}, \\ 0 & \text{if } d_i \text{ is small}. \end{cases}
\]

Considering (17) as a spectral filter method, the associated filter is the same of the Tikhonov method applied to periodic or reflective BCs

\[
\phi_i = \frac{h(\xi_i)^2}{h(\xi_i)^2 + \rho \ell(\xi_i)^2}, \quad i = 1, \ldots, n.
\]

(19)

Exploiting the fact that \( f_j \) and \( f_k \) can be directly computed since the first and last equation of (7) are decoupled, in [16] a suitable first degree polynomial is subtracted from the solution transforming the original problem into one that is solved by means of a discrete sine transform (see Section 4).

In this paper we apply to a generic \( L \) the approach proposed in [16] for \( L = I \) and hence the analysis of \( \mathcal{N}(L) \) could be useful. Considering for instance \( L \) as the finite difference discretization of the second derivative and imposing antireflective BCs, we have

\[
L = A_n(\ell) = \begin{bmatrix} 0 & 2 & \cdots & -1 \\ -1 & 2 & \cdots & -1 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & -1 & 2 \\ 0 & \cdots & 0 & 0 \end{bmatrix}.
\]

(20)

It turns out that \( \dim(\mathcal{N}(L)) = 2 \) and \( \ell(x) = 2 - 2 \cos(x) \) and so in (19) we have \( \phi_1 = \phi_n = 1 \), thus also the reblurring reconstructs exactly the components corresponding to linear functions (indeed they are located at the first and last entries). Therefore, for \( L \) in (20), differently to what happen for \( L = I \), applying the two approaches in [16,18] to Tikhonov regularization, we obtain the same method.

Of course, by changing the regularization technique it is not anymore guaranteed that both approaches yield the same method also with \( L \) as in (20). In fact, in Section 6 we will show that using RSTLS instead of Tikhonov regularization the two strategies yield qualitatively similar results but different reconstructions.
3. RSTLS for simultaneously diagonalizable structures

We describe the method proposed in [11] for solving the RSTLS problem (3) for matrices simultaneously diagonalizable by unitary transforms, i.e., matrices belonging to

$$\delta_U = \{ M : M = U^{\dagger}DU, U^{\dagger}U = UU^{\dagger} = I \}.$$  

We assume that $A, L, E \in \delta_U$. In such a case the RSTLS problem (3) can be decomposed into $n$ one-dimensional minimization problems.

**Theorem 3.1** ([11,27,28]). Suppose that $A, L \in \delta_U$, where $U$ is a given unitary matrix, and let $a, \ell$ be the eigenvalues of $A$ and $L$ defined by the relations

$$UAU^{\dagger} = \text{diag}(a), \quad ULU^{\dagger} = \text{diag}(\ell).$$

Then any solution to the RSTLS problem (3) is given by $f_{\text{reg}} = U^H \hat{f}$, where, for every $i = 1, \ldots, n$, the $i$th component of $\hat{f}, \hat{f}_i$, is an optimal solution to the one-dimensional problem

$$\min_{\tilde{f}_i} \left\{ \frac{|a_i \tilde{f}_i - \hat{g}_i|^2}{1 + \tilde{f}_i^2} + \rho |\tilde{\ell}_i| |\tilde{f}_i|^2 \right\},$$

where $\hat{g} = Ug$. The optimal matrix $E$ is given by

$$E = U^H \text{diag}(r) U,$$

where

$$r_i = \frac{\hat{f}_i (a_i \tilde{f}_i - \hat{g}_i)}{1 + |\tilde{f}_i|^2}.$$  

Although the one-dimensional problems (21) are not unimodal, they can be transformed into (strictly) unimodal problems and consequently solved efficiently and globally. Moreover, the following theorem on the uniqueness of the solution of (3) under standard assumption holds.

**Theorem 3.2** ([11]). Suppose that $A, L \in \delta_U$, where $U$ is a given unitary matrix, and let $a, \ell$ be the eigenvalues of $A$ and $L$ defined by the relations $UAU^{\dagger} = \text{diag}(a)$ and $ULU^{\dagger} = \text{diag}(\ell)$. Let $\hat{g} = Ug$. Then the solution to the RSTLS problem (3) is uniquely attained if and only if for each $i = 1, \ldots, n$ one of the following two conditions is satisfied:

1. $a_i \neq 0$.
2. $a_i = 0$, $\ell_i \neq 0$ and $|\hat{g}_i| \leq \sqrt{|\ell_i|}$.

From Theorem 3.2 it follows that the condition (15) is sufficient for attainment of the optimal solution and necessary for the unique attainment of the optimal solution (see Theorem 4.2 in [11]). On the other hand, differently from the Tikhonov minimization problem (14), the condition (15) is not sufficient for the unique attainment of the optimal solution of RSTLS because the further condition

$$|\hat{g}_i| \leq \sqrt{|\ell_i|}$$

has to be satisfied. The condition (23) is not discussed in [11]. However, we observe that it is usually satisfied for the image deblurring problems when a proper value of the regularization parameter $\rho$ is chosen. Indeed, for image deblurring problems $a_i$ decreases while $|\ell_i|$ increases with $i$, thus $|\ell_i|$ is large when $a_i = 0$. In the noise free case the image deblurring problem satisfies the discrete Picard condition

$$\sum_{i=1}^{n} \left( \frac{\hat{g}_i}{a_i} \right)^2 < +\infty$$

and hence the condition (23) holds. With the addition of noise, the coefficients $\hat{g}_i$ decay (on average) faster than $a_i$, until they level off when the noise in the image starts to dominate the coefficients [1]. Of course, higher is the noise level larger has to be $\rho$ since a greater regularization is required. Therefore, if $\rho$ is properly chosen (e.g., by GCV, L-curve, etc.), than the condition (23) also holds, because the coefficients $\hat{g}_i$ decay (on average) faster than the singular values of the regularized operator also when they level off.

When periodic BCs are applied to (1), in [11] the matrix $U$ is fixed as $U = F^{(n)}$, while if the PSF is symmetric imposing reflective BCs we have $U = C^{(n)}$. Unfortunately, using antireflective BCs the ART (10) is not unitary and so the approach described in this section cannot be directly applied even if the PSF is symmetric.

4. RSTLS for antireflective BCs by discrete sine transform

As briefly sketched in Section 2.3, after having reconstructed the two linear components by using the approach proposed in [16], we can directly apply the method described in Section 3 where the unitary transformation matrix is the discrete sine transform.
Consider the linear equation (1) where the matrix $A$ is the antireflective matrix in (7) and denote by $\hat{f}$ the solution of $A\hat{f} = g$.

From the structure of $A_A$ in (7), $z_1 + h_0 = 1$, the first and the last equations are decoupled, and so
\[
\hat{f}_1 = g_1, \quad \hat{f}_n = g_n.
\]
Let $t_1$ and $t_n$ be the first and the last column of $T_n$ in (10), i.e., $t_1 = \hat{p}/\alpha_n$ and $t_n = Jt_1$. Determine $\beta_1$ and $\beta_2$ such that the first and the last entry of
\[
\hat{f} = \hat{f} - \beta_1 t_1 - \beta_2 t_n
\]
are equal zero, i.e., $\hat{f} = [0, \hat{f}', 0]^T$. By considering the notation in (7), define
\[
g = g - \beta_1 t_1 - \beta_2 t_n
\]
that has the first and the last entry equal zero. Therefore, the linear system
\[
A_A\hat{f} = g
\]
reduces to
\[
\hat{A}\hat{f} = \hat{g},
\]
where $\hat{g} = [0, \hat{g}', 0]^T$. In (8) a regularized solution of (25) was obtained by using the Tikhonov method, while here we use the the RTLS algorithm described in Section 3.

Since $\hat{A}$ is diagonalized by the discrete sine transform (ref. (9)), we compute a regularized solution $\hat{f}_{\text{reg}}$ of (25) solving the following RTLS problem of order $n - 2$ where the involved matrices are diagonalized by $Q^{(n-2)}$. More in detail, let $\hat{A}$ be defined like $\hat{A}$ in (9) where the symbol $h(x)$ is replaced with $\ell(x)$, the Theorem 3.1 can be applied to the RTLS problem
\[
\begin{align*}
&\min_{\hat{f}, \hat{\epsilon}} \| \hat{E} \| + \| \hat{\epsilon} \| + \rho \| \hat{f} \|_2^2 \\
\text{s. t.} \quad &\hat{A} + \hat{E}\hat{f} = \hat{g} + \hat{\epsilon} \\
&\hat{A}, \hat{E} \in \mathcal{S}_Q^{(n-2)}
\end{align*}
\]
obtaining the solution $\hat{f}_{\text{reg}}$.

After having computed $\hat{f}_{\text{reg}}$, a regularized version of $\hat{f}$ can be obtained from (24),
\[
\hat{f}_{\text{reg}} = [0, \hat{f}', 0]^T + \beta_1 t_1 + \beta_2 t_n.
\]
Unfortunately, this approach is not easily extensible to the multi-dimensional case. In the monodimensional case we have only two boundary points which lead to the two coefficients $\beta_1$ and $\beta_2$. In the 2D case, we have to determine $4n - 4$ boundary points, i.e., coefficients, for reducing to an inner problem solvable by 2D sine transforms. This task can be achieved solving four scalar regularization problems in correspondence to the four borders of the image (see [16] for more details). In the 3D case we have to solve eight 2D problems and so on for higher dimensional problems.

5. RTLS for antireflective BCs by ART

Imposing the antireflective BCs the matrix $A$ can be expressed as $A_A = T_n D_n T_n^{-1}$ (ref. (13)). As already said, $T_n$ is not unitary. Therefore, the multiplication of a matrix by $T_n$ does not preserve the spectral norm. Let us introduce the following norm,
\[
\| \hat{f} \| = \| T_n^{-1} \hat{f} \|.
\]
The corresponding induced matrix norm is
\[
\| A \| = \max_{\| \hat{f} \|_1 = 1} \| T_n^{-1} A \hat{f} \| = \max_{\| y \|_1 = 1} \| T_n^{-1} A T_n y \| = \| T_n^{-1} A T_n \|.
\]
For antireflective matrices of the form (13), it holds
\[
\| M \| = \| D \|, \quad \text{if } M = T_n D_n T_n^{-1}.\]
The problem (3) can be reformulated with the norm $\| \cdot \|$ instead of the $\ell_2$-norm as
\[
\begin{align*}
&\min_{\hat{f}, \hat{\epsilon}} \| \hat{E} \| + \| \hat{\epsilon} \| + \rho \| \hat{f} \|_2^2 \\
\text{s. t.} \quad &A + E\hat{f} = g + \epsilon \\
A \text{ and } E \text{ are diagonalizable by } T_n.
\end{align*}
\]
For the new RTLS problem (26) a result analogous to Theorem 3.1 for the RTLS problem (3) holds.
Theorem 5.1. Assume that
\[ A = T_n \text{diag}(a) T_n^{-1} \quad \text{and} \quad L = T_n \text{diag}(\ell) T_n^{-1}. \]

Let \( \hat{g} = T_n^{-1} g \), then any solution to the minimum problem (26) is given by \( \hat{f}_{\text{reg}} = T_n \hat{f} \), where, for every \( i = 1, \ldots, n \), the \( i \)th component of \( \hat{f}, \hat{f}_i \), is an optimal solution to the one-dimensional problem (21). The optimal matrix \( E \) is given by
\[ E = T_n \text{diag}(r) T_n^{-1}, \]
where \( r_i \) is defined in (22).
Proof. The proof is analogous to the one of Theorem 3.1. It is enough to observe that
\[
\min_{\ell, T} \left\{ \| E \|^2 + \| (A + E) f - g \|^2 + \rho \| L f \|^2 \right\}
\]
\[
= \min_{\ell, T} \left\{ \| T_n^{-1} E T_n \|^2 + \| T_n^{-1} (A + E) T_n^{-1} f - g \|^2 + \rho \| T_n^{-1} L T_n^{-1} f \|^2 \right\}
\]
\[
= \min_{r, \hat{f}} \left\{ \| \text{diag}(r) \|^2 + \| \text{diag}(a + r) \hat{f} - \hat{g} \|^2 + \rho \| \text{diag}(l) \hat{f} \|^2 \right\}. \quad \square
\]

As well as for the case of Theorem 3.1 [11], we observe that also here the minimization problem (26) reduces to the same \( n \) one-dimensional minimization problems that we obtain imposing periodic or reflective BCs.

Remark 5.1. Similarly to the cases with periodic or reflective BCs, the problem is reduced to the scalar case. However, the transformed space is the one of the antireflective inverse transform, instead of the space of exponential or cosine transforms. Hence, for the antireflective BCs, the space of the sine frequency functions plus two linear functions is considered. In this case a more accurate reconstruction is obtained in case the signal has a not negligible component in the space of linear functions. We observe that the cosine frequency functions used in the reflective case do not include linear functions.

Theorem 5.2. Let \( A = T_n \text{diag}(a) T_n^{-1}, L = T_n \text{diag}(\ell) T_n^{-1}, \) and \( \hat{g} = T_n^{-1} g. \) Then the solution to the minimization problem (26) is uniquely attained if and only if for each \( i = 1, \ldots, n \) one of the following two conditions is satisfied:

(1) \( a_i \neq 0. \)

(2) \( a_i = 0, \ell_i \neq 0 \) and \( |\hat{g}_i| \leq \sqrt{\rho} |\ell_i|. \)
6. Numerical results for the reconstruction of signals

The numerical experiments are carried out with Matlab 7.0. The Gaussian blur is generated by the function \texttt{psfGauss([k, k], \sigma)} from [1], where the PSF is a \( k \times k \) matrix and \( \sigma \) is the standard deviation. The noise level of the observed signal is defined by

\[
\nu := \frac{\| \epsilon \|}{\| g_0 \|}.
\]
Fig. 7. Relative restoration error varying the regularization parameter $\rho$. The dashed curve is obtained with reflective BCs and the solid curve with antireflective BCs. (a) $\sigma = 5$ and $\nu = 10^{-3}$, (b) $\sigma = 7$ and $\nu = 4 \cdot 10^{-3}$.

where $g_0$ is the noise free blurred image. The Matlab code for RTLS with reflective BCs is available at

http://iew3.technion.ac.il/~becka/papers/rstls_package.zip

from the authors of [11]. The Matlab code for the antireflective transform, its inverse transform, and the computation of the eigenvalues of the antireflective matrix is available at

http://scienze-como.uninsubria.it/mdonatelli/Software/software.html

The two previous codes can be easily combined together to obtain the RTLS with antireflective BCs.

As a first example, we consider a signal blurred with a Gaussian PSF with standard deviation $\sigma = 4.5$ and corrupted by white Gaussian noise with $\nu = 5 \cdot 10^{-3}$ (see Fig. 1). We compare the algorithm in [11] for reflective BCs (det), with our two methods defined in Section 4 (dst) and Section 5 (art). The reconstructed signal taking $L$ equal to the Laplacian and estimations of the standard deviation equal to 4 and 5, is depicted in Figs. 2 and 3, respectively. The reconstructed signal taking $L$ equal to the square of the Laplacian and $\sigma = 4$ is depicted in Fig. 4. The reconstructed signal and the reconstruction error, defined as $\|f - \tilde{f}\|_2/\|f\|_2$, where $\tilde{f}$ is the restored signal, are depicted to the left and to the right, respectively.

We observe that the antireflective BCs yield a better reconstruction in particular close to the boundary with respect to those obtained imposing reflective BCs. The approaches described in Sections 4 and 5 give comparable results. Comparing Figs. 2 and 3, we observe that it is better to underestimate $\sigma$. Indeed, although the reconstruction errors are comparable, the one obtained with $\sigma = 4$ has less oscillations and is sharper with respect to the one obtained with $\sigma = 5$.

An example with the same blur and more noise ($\nu = 2 \cdot 10^{-2}$) is depicted in Fig. 5. Also in this example, the antireflective BCs allow to better reconstruct the signal with respect to the reflective BCs, in particular close to the boundaries of the signal.

7. Image reconstruction

Taking the remarks made in Section 2.3 and the numerical results of Section 6 into account, we consider the technique described in Section 5 for the multi-dimensional case.

7.1. RTLS with 2D antireflective transform

The bidimensional ART can be defined by means of the tensor product,

$$T_n^{(2)} = T_n \otimes T_n.$$  \hspace{1cm} (27)

Therefore, the coefficient matrix of the problem with antireflective BCs can be diagonalized by using $T_n^{(2)}$ obtaining

$$A = A_n(h) = T_n^{(2)} D_n T_n^{(2)},$$

with $h$ a bidimensional trigonometric function depending on the coefficients of the PSF and $D_n$ is a diagonal matrix obtained by sampling the function $h$ in an appropriate way. The description of the structure of the matrix $A_n(h)$ in the multi-dimensional case can be found in [29]. In particular the computation of the eigenvalues of $A_n(h)$ needs to separately consider the points on the boundary of the domain similarly to the approach described in Section 4. Nevertheless, when the codes to compute the ART, its inverse, and the eigenvalues of $A_n(h)$ are available, they can be used in any regularization method based on FFT or DCT simply replacing such transforms with ART.

By using the bi-dimensional ART $T_n^{(2)}$ in (27) the following norm can be defined,

$$\|f\| = \|(T_n^{(2)})^{-1} f\|_2$$

and the same analysis of Section 5 holds just replacing $T_n$ with $T_n^{(2)}$. 


Fig. 8. (a) True image $512 \times 512$, the white box delimit the inner $463 \times 463$ part used in the restoration. (b) Observed image $463 \times 463$, $\sigma = 7$ and $\nu = 4 \cdot 10^{-3}$. (c) Restored image with reflective BCs. (d) Restored image with antireflective BCs. (e) Detail of the restoration with reflective BCs ($200 \times 200$). (f) Detail of the restoration with antireflective BCs ($200 \times 200$).

7.2. Numerical results

We start with the $512 \times 512$ peppers gray image in Fig. 6(a). The blur is generated by Gaussian PSF of dimension $49 \times 49$ with standard deviation $\sigma = 5$ implemented in the function `psfGauss([49, 49], 5)` from [1]. We assume that the blurring is not exactly known and that the PSF is initially approximated by a Gaussian PSF with standard deviation $s = 6$. We then cut the margins by 25 rows and columns resulting with a $463 \times 463$ and add a Gaussian white noise with standard deviation $10^{-3}$ (Fig. 6(b)). The regularization matrix $L$ represent the following discretization of the Laplacian:

$$L = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}.$$
Since in this work we are not interested in strategies for estimating $\rho$, such a parameter is chosen as the minimum of the relative reconstruction error. Of course, this is possible only in synthetic tests where the true solution is known. A better reconstruction, especially close to the border of the image, is obtained with antireflective BCs respect to the one obtained with reflective BCs (see Fig. 6).

The reconstruction errors depending on the regularization parameter $\rho$ are depicted in Fig. 7(a). It can be noticed that the reconstruction error using the antireflective BCs is less than the one obtained with reflective BCs. In particular, the minimum is 0.092 using the antireflective BCs and is 0.097 for the reflective BCs.

We consider now a corrupted image with $\sigma = 7$ and $\nu = 4 \cdot 10^{-3}$. The observed image and the reconstructed ones are depicted in Fig. 8(b)–(f), respectively. As in the previous example, RSTLS with antireflective BCs allows to reduce the ringing effects close to the border of the image at the same computational cost of RSTLS with reflective BCs. The relative reconstruction error is 0.126 with antireflective BCs, it is 0.130 with reflective BCs. The relative reconstruction error, depending on the regularization parameter $\rho$, is depicted in Fig. 7(b).

8. Conclusions

The Regularized Structured Total Least Squares approach has been considered for computing an approximate solution of the linear systems arising in the problem of reconstructing images, given an approximation of the PSF. Although such an approach gives rise to a highly nonconvex optimization problem whose global minimum is difficult to compute, in the paper it is shown that, considering antireflective boundary conditions, the problem can be decomposed into single variable subproblems with an approach similar to that proposed in [11] for periodic and reflective BCs. Such subproblems are then transformed into one-dimensional unimodal real-valued minimization problems which can be solved globally. Some numerical examples show the effectiveness of the proposed approach.

Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions.

References